Finite Element Solution of Corona I-V Characteristics in ESP’s with Multi Discharge Wires

H. Ziedan\(^1\), J. Tlustý\(^2\), A. Mizuno\(^3\), A. Sayed\(^1\), A. Ahmed\(^1\), and R. Procházka\(^2\)

\(^1\)Electrical Engineering Department, Assiut University, Egypt
\(^2\)Faculty of Electrical Engineering, Czech Technical University in Prague, Czech Republic
\(^3\)Department of Environmental and Life Sciences, Toyohashi University of Technology, Japan

Abstract—A finite element method (FEM) has been adopted to model the corona characteristics in duct-type electrostatic precipitators (ESPs) with multi-discharge wires. The study involves the evaluation of the electric potential, field and space-charge in the inter electrode spacing. The calculated current-voltage (I-V) characteristics agreed well with those calculated before based on finite and boundary element methods. Moreover, the I-V characteristics were measured at varying number of discharge wires, wires’ radius and spacing between wires and the collecting plates. The present calculated I-V characteristics agreed well with those measured experimentally in comparison with calculations based on Deutsch’s simplifying assumption.

Keywords—Finite elements, Deutsch’s assumption, corona onset voltage, corona space charge, wire-duct precipitators

I. INTRODUCTION

Electrostatic precipitators (ESPs) are employed in electric power plants and many industries such as cement production, chemical processing and domestic air cleaning. The basic processes of ESP are straight forward and well known [1]. The finite element method (FEM) was introduced by Abdel-Salam et al. [2-6] to solve the ionized field problem for DC transmission lines and wire-duct ESP with one and three discharge wires. Hoburg and others [7, 8] used the FEM to compute iteratively electric potential structure for an assumed charge density distribution in a wire-duct ESP, while the method of characteristic (MOC) was used to compute charge density structure for an assumed electric field distribution. Adamiak [9] calculated the potential and field in wire-duct ESP using the MOC for finding the spatial distribution of the charge density and the boundary element method for solving Poisson’s equation. The surface field of the discharge wires remains constant at the onset value \(E_o\) [2-11]. The onset voltage of corona is a prerequisite for calculating the corona current at a given voltage applied to the discharge wires. The onset voltage of corona in an electrostatic precipitator is influenced by the number of discharge wires, the wires’ radius and the spacing between wires and the collecting plates was investigated before [10, 11].

A method was described [12] for calculating the onset voltage of corona from the discharge wires in duct-type precipitators. The onset voltage was evaluated for wire-duct precipitator. No attention was forwarded [12] to locate the initiation of the corona discharge on the wire surface as the corona is not initiated at the same point on the discharge wires as demonstrated before [10, 11].

Corona current-voltage characteristic in an electrostatic precipitator was influenced by the number of discharge wires, the wires’ radius and the spacing between wires and the collecting plates [13]. The corona current-voltage characteristics were calculated for each wire by simultaneously solving Poisson, current density and current continuity equations based on some simplifying assumptions. The most important one is Deutsch’s assumption where the space charge affects only the magnitude not the direction of the electric field [13].

This paper is aimed at calculating the corona current-voltage (I-V) characteristics of wire-duct precipitators with 1, 3, 5 and 7 discharge wires using finite element method (FEM). The accuracy of the calculation method is checked by comparing the theoretical predictions with the corresponding measured values. The method is supported by an iterative procedure to set a guess for the initial distribution of the charge density around the periphery of the discharge wire(s). The iterative procedure is based on simultaneous solution of Poisson, current density and current continuity equations as done before [13]. The change of the corona onset voltage around the periphery of the discharge wires is assessed and considered in the iterative procedure. The calculated corona I-V characteristics for ESPs with 1, 3, 5 and 7 discharge wires are closer to previous measurements [6] and present measurements than previous calculation [6, 9].

II. GOVERNING EQUATIONS

Poisson’s equation relates the electric field intensity \(E\) to the space charge density \(\rho\) within precipitator volume as:

\[
\nabla \cdot E = \frac{\rho}{\varepsilon_0}
\]

(1)
where \( \varepsilon_0 \) is the permittivity of free space. At the same time, the continuity of the current density vector \( \mathbf{J} \) is homogeneous:

\[
\nabla \cdot \mathbf{J} = 0
\]

The current density is determined by the electric field as:

\[
\mathbf{J} = k \mathbf{\rho} \mathbf{E}
\]

where \( k \) is the mobility of the ions. Defining the scalar potential \( \phi \), which is related to the electric field intensity by:

\[
\mathbf{E} = -\nabla \phi
\]

All attempts in the literature for simultaneous solutions of Eqn. (1) through Eqn. (3) in ESPs were based on some simplifying assumptions [3-13]. The most common ones are as follows:

(a) The electrode spacing is filled with unipolar space charge like wires’ polarity.
(b) The mobility of ions is constant.
(c) Thermal diffusion is neglected.
(d) The surface field of the discharge wires remains constant at the onset value \( E_0 \).
(e) The discharge wires and collecting plates are long enough along \( z \)-direction, so Eqn. (1) in the 2-dimensional, \( x \)- and \( y \)-coordinates system is expressed as:

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\mathbf{\rho}}{\varepsilon_0}
\]

III. METHOD OF ANALYSIS

A. Finite Element Grid

The geometry of wire-duct ESPs of one- and multi-discharge wires is shown in Fig. 1. The voltage on the discharge wires is the applied voltage \( V \), whereas the two collecting plates are grounded, i.e. \( V = 0 \). The magnitude of electric field at the surface of the discharge wires is assumed to be constant at the onset value, \( E_0 \), but change from point to point around the wire periphery as calculated before [10, 11]. The field lines extending
between discharge wires and collecting plates divide the precipitator volume into flux tubes, along them the ions conduct from wires to collecting plates. The FE grid is generated from quadrangles produced by the intersection of field lines with equi-potential contours, Fig. 2. Each quadrangle is divided into two triangle elements. In this way, the intersection nodes constitute the vertices of the elements forming the finite-element grid. The first grid is mapped in the absence of space charge using the accurate charge simulation method [10-15] as described in Appendix (1). The intersection of the $ith$ flux line and the $jth$ equi-potential contour represents the node ($i, j$) of the proposed grid, Fig. 2. The values of the nodal field and potential are denoted as $E_{j}^{(i)}$ and $\phi_{j}^{(i)}$, that is, the first estimated values at the grid nodes. The surface space-charge density located at node ($i, 1$) around the periphery of the wires is obtained by an iterative procedure based on simultaneous solution of Poisson, current density and current continuity equations [13] as described in Appendix (2).

Using Eqn. (3), the equation of continuity of current density is expressed as:

$$\nabla \cdot (k \rho E) = 0$$  \hspace{1cm} (6)

Along each flux-tube:

$$\frac{d \rho}{dt} = -\frac{\rho^2}{\varepsilon_o E}$$ \hspace{1cm} (7)

where the ion mobility is assumed constant. The unit vector $t$ is assumed along the axis of the flux-tube or along the direction of the electric field $E$. Starting at the discharge wire surface, integration of Eqn. (7) gives the space-charge density at all nodes located along the axis of the flux-tube.

**B. Finite Element Solution of Poisson’s Equation**

The potential $\phi$ within each element is approximated as a linear function of coordinates [16], namely:

$$\phi = \phi_0 W^er + \phi_1 W^e + \phi_2 W^e$$ \hspace{1cm} (8)

with $\rho, s$ and $t$ represented the nodes of the element $e$, Fig. 2, and $W$ is the corresponding space shape function.

To solve Poisson’s equation (5), in the general form, an energy functional $RF^{(0)}$ expressed by the following equation is formulated [16]:

$$RF^{(0)} = \int_A [W^T \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 + \frac{\rho}{\varepsilon_o}] dA$$ \hspace{1cm} (9)

where $A$ is the area of triangle element, $[W]$ is the row vector containing the element’s shape function and $[W]^T$ is the transposed of $[W]$. For known values of $\rho$ at nodes, Poisson’s equation is solved by minimizing the energy functional $RF^{(0)}$ with respect to each nodal potential value. This minimization leads to a set of simultaneous equations of $\phi$ at the nodes in the form:

$$[K][\phi] = [F]$$ \hspace{1cm} (10)

where $[K]$ is the element stiffness matrix, $[\phi]$ is the unknown potentials of the element nodes, $[F]$ is the free term which entails the charge density values at the element nodes as described in Appendix (3).

On applying Eqn. (10) for all elements of the grid, the following set of equations is obtained [16]:

$$[K][\phi] = [F]$$ \hspace{1cm} (11)

where $[K]$ is the global stiffness matrix $= \sum_{e=1}^{W} [K^e]$  

$[\phi]$ is the estimated nodal potentials vector

$[F]$ is the assembled free term due to boundary conditions, $= \sum_{e=1}^{W} [F^e]$

Solution of the set (11) determines the array of nodal potential $\phi$. This is the second estimate of the nodal potentials $\phi_{(2)}$ due to the presence of the space charge. The constancy of the wire surface field at $E_0$ is to be implemented into FE formulation. This is achieved by noting that $(\phi_{(1)} - \phi_{(2)})/\Delta r_i = E_0$ where $\Delta r_i$ is the radial distance between the first two nodes along the axis of the $ith$ flux-tube, $\Delta r_i$ is chosen much smaller than the radius of the discharge wire, Fig. 2. Since the potential $\phi_{(1)}$ is the applied voltage which is known, then $\phi_{(2)}$, the potential at node ($i, 2$), is also of known potential for the $ith$ flux-tube.

**C. Potential Updating**

The last two estimates of the potential at each node, $\phi_{(0)}$ and $\phi_{(n+1)}$, are compared. A nodal potential error $E_v$ relative to the average value of the nodal potential is defined as [5]:

$$E_v = \frac{\phi_{(n)} - \phi_{(n+1)}}{(\phi_{(n)} + \phi_{(n+1)})/2}$$ \hspace{1cm} (12)

If the maximum nodal potential error exceeds a prescribed value $\delta_v$, a correction of $\rho_{(1)}$, which is the space charge density on wire surface, was made as [5]:

$$\rho_{(1)_{new}} = \rho_{(1)_{old}}[1 + \tau \cdot \text{max} (E_v)], i=1, 2, 3, \ldots, M$$ \hspace{1cm} (13)

where $\tau$ is an acceleration factor, $\text{max} (E_v)$ is the maximum nodal potential error and $M$ is the number of flux lines in the considered region. The space-charge
Input data: \( r_c, d, H \) and \( N \)

Calculate potential and field at each point of ESP volume in absence of space-charge using Charge Simulation Method with simulation charges \( Q_s \), Appendix (1).

Mapping field-lines and equi-potential contours to form first grid.

Calculate surface-space-charge density \( \rho_{i,j} \) on wire surface, Appendix (2).

Integration of Eqn. 7 for evaluation of the space-charge density \( \rho_{i,j} \) at each node of the grid.

Finite element solutions of Poisson’s equation to obtain a new estimate of nodal potentials due to presence of space-charge.

Update \( \rho_{i,1} \) at the wire’s surface.

Is nodal potential error, \( E_v < \delta_1 \) \( (= 10^{-6}) \)?

Yes

Calculate charge per unit length \( Q_{v,i,j} \) for each node using Eqn. 14.

Mapping of new field lines and equi-potential contours due to \( Q_s \) and \( Q_v \) to form a new grid

No

Corona current calculation using Eqn. 15

Is maximum nodal space-charge density mismatch between two grids < \( \delta_2 \) \( (=10^{-6}) \)?

Yes

Calculate new nodal potential and field values

No

End

Fig. 3. Procedure of calculations flow chart.
density at other nodes along all flux-tubes are updated by integrating Eqn. (7) to keep the continuity condition of current satisfied. Using the estimated nodal space-charge densities, the FE algorithm is applied again to obtain a new estimate of the nodal potentials [3]. Iteration of the space-charge correction and potential estimation continues until the nodal potential error is less than $\delta_1$, Fig. 3, a prescribed error value determined by the required accuracy of the FE solution.

**D. Grid Updating**

The space charges distributed over the precipitator volume are represented by discrete line charges at the grid nodes. Hence, the charge per unit length, $Q_{vij}$, at node $(i, j)$ is determined by the following equation [5]:

$$Q_{vij} = \rho_{ij} v_{ij}$$  \hspace{1cm} (14)

where $v_{ij}$ is the volume per unit length surrounding the node $(i, j)$ as shown in Fig. 4 and $\rho_{ij}$ is the corrected charge density at the same node. The lines defining the volume are also field lines emanates from the discharge wire surface and terminates at the collecting plate. The angle, at which the volume bounding line emanates, lies between of two successive field lines. Intersection of these lines with equipotential contours generates the volume surrounding each node of the grid as shown in Fig. 4.

The potential and field are calculated due to both the simulation charges $Q_s$ resulting from the applied voltage and the simulation charges $Q_v$ due to the corona ions filling the precipitator volume. New field lines and equipotential contours are mapped to form a new grid.

The procedure of grid updating is repeated until the grid assumes a stationary structure. This can be measured by having the maximum nodal space-charge density mismatch between the last two grids less than a prescribed value $\delta_2$, Fig. 3.

**E. Corona Current Characteristics**

The procedure of potential and grid updating (sections III.C and III.D) is repeated until the solution reaches a steady state at a given applied voltage $V$ above the onset value. Then, the corona current of each discharge wire, is equal to the sum of the currents flowing through the flux tubes emanating from the wire. The corona current density $J_{en}$ at the surface of the discharge wire where the nth flux-line extending along the axis of the nth flux tube is obtained using eqn. (3), i.e.,

$$J_{en} = k \rho_{en} E_{0n}$$  \hspace{1cm} (3a)

where $\rho_{en}$ and $E_{0n}$ correspond to the values of and for the nth flux line, Appendix (2).

The corona current density $J$ averaged over the wire periphery is calculated as:

$$J = \left( \sum_{n=1}^{M} J_{en} \right) / M$$  \hspace{1cm} (15)

where $M$ represents the flux tubes emanating from the discharge wire. Multiplying the average value of current density obtained by Eqn. (15) by the surface area per unit length of the wire yields the value of the wire corona current. Summation of the corona current per wire for all ESP wires determines the corona current $I$ of the precipitator at the applied voltage value $V$.

**IV. CALCULATED CORONA CURRENT-VOLTAGE CHARACTERISTICS AGAINST PREVIOUS CALCULATIONS**

The maximum nodal potential error $\delta_1$, the acceleration factor $\tau$ and the maximum space-charge density mismatch between the last two grids $\delta_2$ were taken equal to $10^{-6}$, 0.2 and $10^{-6}$, respectively.

The current-voltage characteristic for one-wire ESP (wire radius = 1 mm and wire-to-plate spacing = 11.43 cm) is shown in Fig. 5. It is clear from this figure that the proposed method predicts current values are very close to those obtained before by Abdel-Salam et al. using finite elements [6] and Adamiak using boundary elements [9] at the same applied voltages. However, there is a deviation between the present calculated current values and those obtained based on Deutsch’s assumption [13]. This is attributed to the approximation imposed on the calculation procedure due to the use of Deutsch’s assumption.

The proposed method is also applied for calculating the current-voltage characteristics of a one-wire ESP (wire radius = 0.5 mm and wire-to-wire spacing = 13 and 23 cm), shown in Fig. 6, to compare with Abdel-Salam’s measurements [6]. It is quite satisfactory that the calculated corona current-voltage characteristics by FEM agreed well with those obtained experimentally [6].
Again, there is a deviation between the present calculated current values and those obtained based on Deutsch’s assumption [13].

V. EXPERIMENTAL CHECK-UP OF CALCULATED CORONA CURRENT-VOLTAGE CHARACTERISTICS

A. Experimental Set-Up

A wire-duct electrostatic precipitator (ESP) was set-up in the HV laboratory of Czech Technical University in Prague, Czech Republic to compare the calculated and measured corona current-voltage characteristics for different ESP configurations. The details of the set-up were described elsewhere [13].

B. Experimental Technique

To measure the corona current-voltage characteristics from each discharge wire, a shielded micro-ammeter is connected to the discharge wire and the corona current is recorded as described above with the increase of the applied voltage.

The corona current-voltage characteristics were measured for each discharge wire of ESP. The corona currents of each discharge wires were measured for different voltage values applied to the discharge wires. Different values of wire radius ($r_c = 0.26$, $0.935$ and $1.975$ mm) were used for ESPs with $1, 3, 5$ and $7$ discharge wires and plate-to-plate spacings ($2H$) adjusted to $30$ and $40$ cm.

All the measurements are made in HV laboratory of pressure = 1001.3 kPa and temperature = 22°C.
The maximum voltage applied to the set-up circuit without flash-over between HV feeding terminals and steel supporting frame is about 120 kV. Therefore, the applied voltage can be increased safely up to 110 kV.

C. Calculated and measured corona current-voltage characteristics

Corona current-voltage characteristics were measured and calculated using FEM for wire-duct precipitators with different numbers of discharge wires at varying wire radius and plate-to-plate spacing as shown in Figs. 7-13. The wire-to-wire spacing was kept constant at 14.5 cm. It is satisfying that the calculated characteristics by FEM agreed reasonably - within the experimental scatter - with those measured experimentally better than previous calculations [13] based on Deutsch’s assumption.

Effect of wire radius ($r_c$): The smaller the wire radius, the smaller the onset voltage and the higher is the corona current at the same applied voltage and plate-to-plate spacing for a single-wire precipitator, Fig. 7. Also, the smaller the wire radius, the smaller the onset voltage and the higher is the central-wire corona current at the same applied voltage and plate-to-plate spacing for 3-wire precipitators, Fig. 8, 5-wire precipitators, Fig. 9 and 7-wire precipitators, Fig. 10. The decrease of the corona onset voltage with the decrease of the wire radius is attributed to the corresponding field enhancement at the wire surface [10, 11]. The corona current depends on how high the applied voltage above the onset value. This is why the corona current at the same applied voltage increases with the decrease of the onset voltage.

For the same wire radius ($r_c$), the central-wire corona current decreases with the increase of the plate-to-plate spacing ($2H$) at the same applied voltage. This is simply attributed to the resulting decrease of the electric field along the flux lines where the ions convict to conduct the corona current.
Effect of plate-to-plate spacing ($2H$): The larger the plate-to-plate spacing, the higher the onset voltage and the smaller is the corona current at the same applied voltage and wire radius for a single-wire precipitator, Fig. 7. Also, the larger the plate-to-plate spacing, the higher the onset voltage and the smaller is the central-wire corona current at the same applied voltage and wire radius for 3-wire precipitators, Fig. 8, 5-wire precipitators, Fig. 9 and 7-wire precipitators, Fig. 10. It is quite clear that the corona current decreases with the increase of plate-to-plate spacing, Figs. 7-10, for the same applied voltage and wire radius. This is simply explained by the resulting decrease of the electric field along the flux lines, where the corona ions are convicting between the discharge wires and the collecting plates. The more of plate-to-plate spacing the more is the decrease of the electric field along the flux lines with subsequent decrease of the corona current for the same applied voltage in agreement with Figs. 7-10.

Effect of number of discharge wires ($N$): The larger the number of discharge wires, the more is the shielding effect imposed on the central wire with a subsequent increase of the corona onset voltage and decrease of corona current emitted from the central wire at the same applied voltage and plate-to-plate spacing, Figs. 7-10 for $N = 1, 3, 5$ and 7.

Fig. 11 shows the corona current-voltage characteristics of central and outer wires of 3-wire precipitator for the same wire radius and plate-to-plate spacing. Fig. 12 shows the corona current-voltage characteristics of central and the two outer wires to the right of the central one of 5-wire precipitator for the same
Fig. 11. I-V characteristics of central- and outer- wires of three-wire ESP ($r_c = 0.935$ mm, $d = 14.5$ cm, $2H = 30$ cm).

Fig. 12. I-V characteristics of central-, R1- and outer- (R2) wires of five-wire ESP ($r_c = 0.935$ mm, $d = 14.5$ cm, $2H = 30$ cm).

Fig. 13. I-V characteristics of central-, R1-, R2- and outer- (R3) wires of seven-wire ESP ($r_c = 0.935$ mm, $d = 14.5$ cm, $2H = 30$ cm).
VI. CONCLUSIONS

(1) Corona current-voltage characteristics of a laboratory electrostatic precipitator with 1, 3, 5 and 7 discharge wires are calculated based on FEM and measured for each wire in the laboratory. The calculated values agreed reasonably with those measured experimentally.

(2) The calculated corona current-voltage characteristics by FEM agreed well with those obtained experimentally in comparison with calculations based on Deutsch's assumption [13].

(3) The smaller the wire radius, the smaller the onset voltage and the higher is the corona current at the same applied voltage and wire radius for multi-discharge precipitators irrespective of the number of discharge wires.

(4) The larger the plate-to-plate spacing, the higher the onset voltage and the smaller is the corona current at the same applied voltage and wire radius for multi-discharge precipitators irrespective of the number of discharge wires.

(5) The larger the number of discharge wires, the more is the shielding effect imposed on the central wire with a subsequent increase of the corona onset voltage and decrease of corona current emitted from the central wire at the same applied voltage and plate-to-plate spacing.

(6) The current per discharge wire in a multi-discharge precipitator increases toward the end of the collecting plates for the same applied voltage and same radius of the discharge wires.

ACKNOWLEDGMENT

The authors would like to thank Prof. Mazen Abdel-Salam of Assiut University, Egypt for his interest in this research work.

REFERENCES


APPENDICES

A. Charge-Simulation Technique for Calculating Space-Charge-Free Field in Wire-Duct ESPs

This technique was used for calculating the electric field for wire-duct ESP with \( m \) wires (\( m \) is odd), [10]. Fig. 14 shows a one-quadrant of the cross-section of the ESP in the \( x-y \) plane. The surface charge on each wire is simulated by \( (N_1) \) line charges located at radius \( r_f \), where \( r_f = f \times r_c \), where \( f \) is a fraction, chosen 0.5 in the present work and \( r_c \) is the radius of discharge wire. The surface charge on each plate of the ESP is simulated by a number \( (N_2) \) line charges located outside the plate at a distance from the plate (a) equal to the distance between two adjacent simulation charges (b).

Thus, the total number of unknown simulation charges is \( (mN_1 + 2N_2) \). There is a symmetry around both \( x- \) and \( y- \) axes, Fig. 14, which reduces the number of unknowns to \( n = (mN_1 + 2N_2)/4 \).

To evaluate the unknown simulation charges \( Q_{sj} \), \( j = 1, 2, 3, \ldots, n \), a set of boundary points equal to the simulation charges is chosen on the surface of discharge wires and collecting plates as shown in Fig. 16 to satisfy the boundary conditions:

- \( \phi = V \) at discharge wires.
- \( \phi = 0 \) at collecting plates.

The potential \( \phi \) is at the \( i \)th boundary point of coordinates \((x_i, y_i)\) is the sum of potential contributions due to all simulation charges.

\[
\Phi_i = \left( \frac{1}{2\pi \epsilon_0} \right) \sum_{j=1}^{n} \ln \left( \frac{1}{R_1 R_2 R_3 R_4} \right) Q_{ij} \quad (A-1)
\]

where,

\[
R_i = (x_i - x_j)^2 + (y_i - y_j)^2
\]

The magnitude of the electric field intensity at point \( p \) is calculated as:

\[
\xi_p = \sqrt{\xi_x^2 + \xi_y^2} \quad (A-4)
\]

B. Electric field and space charge distributions

Based on Deutsch’s assumption where the space charge affects only the magnitude not the direction of the electric field [13], one can write i.e.,

\[
R_2^2 = (x_i + x_j)^2 + (y_i - y_j)^2
\]

\[
R_3^2 = (x_i + x_j)^2 + (y_i + y_j)^2
\]

\[
R_4^2 = (x_i - x_j)^2 + (y_i + y_j)^2
\]

\((x_i, y_i)\) are the coordinates of the \( j \)th simulation charge.

Satisfaction of the boundary conditions at the boundary points formulates a set of equations relating the values of simulation charges to the potential values at the boundary points.

The solution of this set determines the unknown simulation charges \( Q_{sj} \).

Once the simulation charges are known, the electric field intensity at any point \( p \) \((x_p, y_p)\) can be determined:

\[
\xi_i = \left( \frac{1}{2\pi \epsilon_0} \right) \sum_{j=1}^{n} Q_{ij} \left( (x_i - x_j \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + (y_i + y_j \left( \frac{1}{R_3} + \frac{1}{R_4} \right) \right) \right)
\]

(A-2)

\[
\xi_j = \left( \frac{1}{2\pi \epsilon_0} \right) \sum_{i=1}^{n} Q_{ij} \left( (y_i - y_j \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + (x_i + x_j \left( \frac{1}{R_3} + \frac{1}{R_4} \right) \right) \right)
\]

(A-3)

Fig. 14. Arrangement of simulation charges and boundary points for one quarter of a wire duct ESP.
\[
E = \lambda \zeta \quad \text{(A-5)}
\]

where \( \lambda \) is a scalar point function of space coordinates depending on charge distribution and \( \zeta \) is the space-charge-free field. The calculation of the space-charge-free-field within the inter-electrode spacing between the discharge wires and the collecting plates was reported before [11] as explained in Appendix A.

Mathematical manipulation of Eqns (1) through (4) and Eqn. (A-5) ends up with the following equations for the distributions of the charge density \( \rho \) and the scalar point function \( \lambda \) along the axis of the flux tubes between the discharge wires and the collecting plates:

\[
\frac{1}{\rho^2} = \frac{1}{\rho_e^2} + \frac{2}{\varepsilon_o \rho_e \lambda_e} \int d\phi \xi^2 \quad \text{(A-6)}
\]

\[
\lambda^2 = \lambda_e^2 + \frac{2 \rho_e \lambda_e}{\varepsilon_o \rho_e} \int d\phi \xi^2 \quad \text{(A-7)}
\]

where \( \rho_e \) and \( \lambda_e (= V_0/V) \) are the values of \( \rho \) and \( \lambda \) at the surface of discharge wires. \( V \) and \( V_0 \) are the applied and corona-onset voltages.

The procedure for determining the value of \( \rho_e \) at the surface of the discharge wire where a flux line emanates was explained before [13].

C. Free term accommodating element boundary conditions

To calculate \([f^e]\) for any element,

\[
[f^e] = -A \begin{pmatrix} \rho_p \\ \rho_s \\ \rho_t \end{pmatrix} \quad \text{(A-8)}
\]

where \( \rho_p, \rho_s \) and \( \rho_t \) are space charge density at nodes \( p, s \) and \( t \) of element \( e \), respectively, Fig. 2. \( A \) is the per unit length area of the triangle in the \( z \)-direction as determined as follows:

\[
2A = \begin{vmatrix} 1 & x_p & y_p \\ 1 & x_s & y_s \\ 1 & x_t & y_t \end{vmatrix} \quad \text{(A-9)}
\]

\((x_i, y_i), i = p, s \) and \( t \), are the nodal coordinates of element \( e \).